

STA 111: Probability & Statistical Inference

Lecture Eight – Functions of Random Variables D.S. Sections 3.8 and 3.9

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Outline

- Questions from Last Lecture
- Functions of Continuous Random Variables
- Cumulative Distribution Function (CDF) Technique
- Change of Variable Technique
- Recap

Objectives

By the end of class, you should be able to derive the marginal distribution of functions of a continuous random variable using the:

- Cumulative Distribution Function (CDF) Technique
- Change of Variable Technique

Motivation

- Let X be a continuous random variable, and $Y = h(X)$, a function of X . We have already talked about how to find the expectation of Y from previous lectures. What if we want more than that?
- For simplicity, we will assume that $h(X)$ is a well-behaved function of X . That is, h is continuous, monotone and its inverse h^{-1} exists.
- Notice that Y must also be a random variable that has its own distribution since it is a transformation of a random variable. Can we derive that?
- *Example 1:* Let X be a continuous random variable with pdf $f_X(x) = 4x^3$, $0 < x < 1$ and let $Y = X^2$. What is the pdf of Y ?

Illustration

- Sometimes it makes more sense (and can be relatively straightforward) to first find the cdf of Y and then derive the pdf using calculus and what we have learned so far.
- Back to our example, what is the cdf of X ? [We'll do that in class.](#)
- Recall: $f_X(x) = 4x^3$, $0 < x < 1$.

Illustration

– Then, $F_Y(y) = P(Y \leq y) = P(X \leq \sqrt[4]{y}) = F_X(\sqrt[4]{y}) = (\sqrt[4]{y})^4 = y^2$

Do you know why $X \not\leq \sqrt{y}$?

– Now that we know what the cdf of Y is, can you figure out what the cdf of X is?

– What is the support for y ?

Another Illustration

Example 2: Let X_1, X_2, \dots, X_n be a random independent and identically distributed (iid) sample from the distribution with pdf $f(x) = 3x^2$ on $0 \leq x \leq 1$

Now, Let $Y = \max\{X_1, X_2, \dots, X_n\}$. Can we find $E(Y)$?

First, can we figure out what the pdf of Y is?

(Think about this: what does being the maximum really mean? If the maximum of X random variables is less than y , what is the relationship between each of the X 's and y ?)

Also, what is the cdf of X ?

Another Illustration (Cont'd)

As in the previous example, first find the cdf $F_X(x)$ of X .

$$\text{Thus, } F(y) = P(Y \leq y) = P(X_1 \leq y, X_2 \leq y, \dots, X_n \leq y)$$

$$\Rightarrow P(Y \leq y) = P(X_1 \leq y \text{ and } X_2 \leq y \text{ and } \dots, \text{ and } X_n \leq y)$$

$$\Rightarrow P(Y \leq y) = P(X_1 \leq y) \times P(X_2 \leq y) \times \dots \times P(X_n \leq y)$$

Can you see why this is the case?

$$\Rightarrow F_Y(y) = P(Y \leq y) = F_{X_1}(y) \times F_{X_2}(y) \times \dots \times F_{X_n}(y)$$

$$\Rightarrow F_Y(y) = y^3 \times y^3 \times \dots \times y^3 = y^{3n}$$

Another Illustration (Cont'd)

$$\Rightarrow f_Y(y) = (3n)y^{3n-1} \text{ on } 0 \leq y \leq 1$$

Do you know why that is the support of Y ?

Now,

$$E(Y) = \int_0^1 y(3n)y^{3n-1} dy = 3n \int_0^1 y^{3n} dy$$

$$\text{Therefore, } E(Y) = \frac{3n}{3n+1}$$

How about $Z = \min\{X_1, X_2, \dots, X_n\}$, can you find $E(Z)$? *To be done in class.*

Shortcut to the cdf technique

If the function $h(X)$ is a continuous function with an inverse (which we already assumed), there is a neater and faster way to find its pdf.

Let $Y = h(X)$, then if the inverse exists, $X = h^{-1}(Y)$.

Also, the derivative of X with respect to Y is $\frac{dX}{dY}$

Thus, the pdf of Y , $f_Y(y) = f_X[h^{-1}(Y)] \times \left| \frac{dX}{dY} \right|$

Example 1 again

$f_X(x) = 4x^3$ on $0 \leq x \leq 1$ and $Y = X^2$.

Inverse of $Y = X^2$ is $X = h^{-1}(Y) = \sqrt[4]{y}$ and $\frac{dX}{dY} = \frac{1}{2\sqrt[4]{y}}$

Then, $f_X[h^{-1}(Y)] = f_X[\sqrt[4]{y}] = 4(\sqrt[4]{y})^3 = 4y(\sqrt[4]{y})$

Therefore, $f_Y(y) = \frac{1}{2\sqrt[4]{y}} \times 4y(\sqrt[4]{y}) = 2y$

Do we have the same answer as before?

Another Example

Example 3 (D.S. Chapter 3 Exercises Question 4 – to be done in class):

Suppose that the pdf of a random variable X is $\frac{x}{2}$ for $0 \leq x \leq 2$. What is the cdf of $Y = 4 - X^3$? We'll try both methods!

Recap

You should now be able to:

- Derive the marginal distribution of functions of a continuous random variable using the CDF method
- Use the change of variable technique as a shortcut to the cdf method